Improved Cryptanalysis of Rank Metric Schemes Based on Gabidulin Codes

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Linear code

1. Linear code = vector space over a finite field

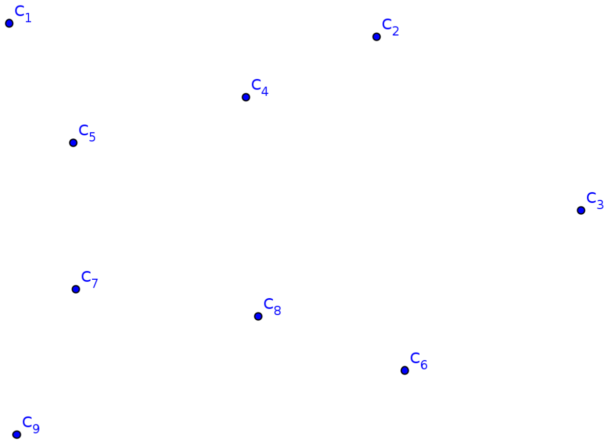
$$\mathscr{C} = \bigoplus_{i=1}^{k} \mathbb{F}_q \, \vec{v}_i$$

   where $\vec{v}_i$ are linearly independent.

2. Any $k \times n$ matrix $\boldsymbol{G}$ whose rows form a basis of $\mathscr{C}$ is a generator matrix of $\mathscr{C}$.

3. Decoding a word $\vec{w} \in \mathbb{F}_q^n$ = Closest Vector Problem (CVP).

Introduction



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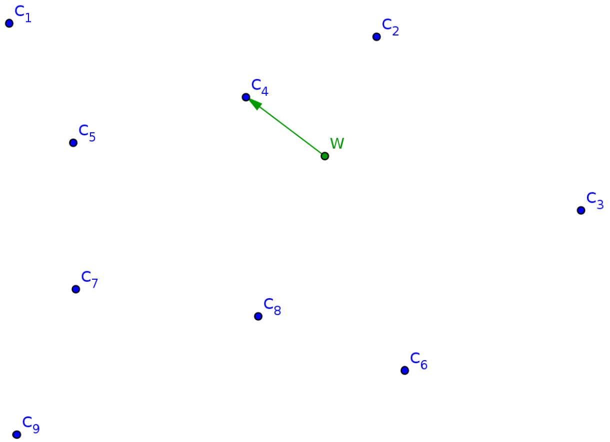

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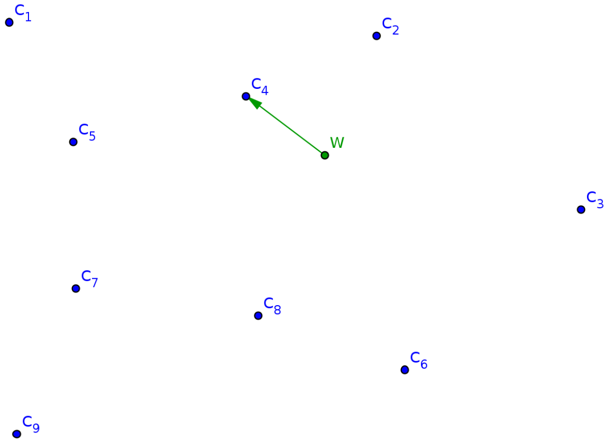

- Decoding is NP-Hard for a random linear code (Berlekamp-McEliece-Van Tilborg '78)

McEliece Public-Key Encryption Scheme ('78)

1. Use code in Hamming metric
2. Based on linear codes equipped with an efficient decoding algorithm
   - Public key = random basis
   - Private key = decoding algorithm
3. McEliece proposed binary Goppa codes
   - No efficient attack on the system up to now
   - Problem of huge key size

McEliece Variants

1. Use another family of code
2. Use another metric instead of Hamming metric

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GPT cryptosystem '91

1. Rank metric with Gabidulin codes
2. But many attacks
   - Gibson's attacks '95, '96
   - Overbeck's attack '05

Some GPT Variants

- **Gabidulin '08**
- **Rashwan-Gabidulin-Honary, '10**

Purpose of this presentation

Polynomial attack against above reparations

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Outline

Plan

Definition 1

- n, m and q are integers with $n \leqslant m$
- $\mathbb{F}_{q^m} = \mathbb{F}_q < w >$
- $\mathscr{B} = \{b_1, b_2, \ldots, b_m\}$ a $\mathbb{F}_q-$basis of $\mathbb{F}_{q^m}$

We define the one to one application ϕ by:

$$\begin{array}{cccc} \phi : & \mathbb{F}_{q^m} & \longrightarrow & \mathcal{M}_{m\times 1}(\mathbb{F}_q) \\ & x = \sum_{i=1}^{m} x_i b_i & \longmapsto & \phi(x) \stackrel{\text{def}}{=} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_m \end{pmatrix} \end{array}$$

Extension Of ϕ

- For a vector $\vec{x} = (x_1, x_2, ..., x_n) \in \mathbb{F}_{q^m}^n$

$$\phi(\vec{x}) \stackrel{\text{def}}{=} (\phi(x_1), \phi(x_2), ..., \phi(x_n)) \in \mathcal{M}_{m \times n}(\mathbb{F}_q)$$

- And for a matrix $\boldsymbol{M} = (m_{ij}) \in \mathcal{M}_{k \times \ell}(\mathbb{F}_{q^m})$

$$\phi(\boldsymbol{M}) \stackrel{\text{def}}{=} (\phi(m_{ij})) \in \mathcal{M}_{km \times \ell}(\mathbb{F}_q)$$

Definition 2 (Rank Weight)

- $\vec{x} = (x_1, x_2, ..., x_n) \in \mathbb{F}_{q^m}^n$

The rank weight of $\vec{x}$ is defined by

$$\|\vec{x}\|_q \stackrel{\text{def}}{=} Rank\,(\phi\,(\vec{x}))$$

Example

Let $\mathbb{K} = \mathbb{F}_{2^5} = \mathbb{F}_2 < w >$, $\vec{x}_1 = (w, w, w, w, w)$, $\vec{x}_2 = (1, w, w^2, 1 + w^3, w^4)$ and $\mathscr{B} = \{1, w, w^2, w^3, w^4\}$.

- $$\phi(\vec{x}_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \phi(\vec{x}_2) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- So,

$$\|\vec{x}_1\|_q = 1, \quad \|\vec{x}_2\|_q = 5$$

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Preliminaries

Definition 2 (Rank Weight)

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- So,

$$\|\vec{x}_1\|_q = 1, \quad \|\vec{x}_2\|_q = 5$$

Lemma 3

- $\vec{x} \in \mathbb{F}_{q^m}^n$
- $\boldsymbol{T} \in \mathrm{GL}_n(\mathbb{F}_q)$

$$\|\vec{x}\boldsymbol{T}\|_q = \|\vec{x}\|_q$$

Definition 4

For $\boldsymbol{M} \in \mathcal{M}_{k \times n}(\mathbb{F}_{q^m})$, the rank of $\boldsymbol{M}$ over $\mathbb{F}_q$ will denotes:

$$\mathtt{Rank}_{\mathbb{F}_q}(\boldsymbol{M}) \stackrel{\text{def}}{=} Rank\,(\phi(\boldsymbol{M}))$$

Definition 5 (Gabidulin codes)

- $\vec{g} \in \mathbb{F}_{q^m}^n$ with $\|\vec{g}\|_q = n$

The $(n,k)-$Gabidulin code $\mathscr{G}_k(\vec{g})$ is the code generated by:

$$\boldsymbol{G} = \begin{pmatrix} g_1^{[0]} & g_2^{[0]} & \cdot & \cdot & \cdot & g_n^{[0]} \\ g_1^{[1]} & g_2^{[1]} & \cdot & \cdot & \cdot & g_n^{[1]} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ g_1^{[k-1]} & g_2^{[k-1]} & \cdot & \cdot & \cdot & g_n^{[k-1]} \end{pmatrix} = \begin{pmatrix} g_1^{q^0} & g_2^{q^0} & \cdot & \cdot & \cdot & g_n^{q^0} \\ g_1^{q^1} & g_2^{q^1} & \cdot & \cdot & \cdot & g_n^{q^1} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ g_1^{q^{k-1}} & g_2^{q^{k-1}} & \cdot & \cdot & \cdot & g_n^{q^{k-1}} \end{pmatrix}$$

$\vec{g}$ is called generator vector of $\mathscr{G}_k(\vec{g})$.

Proposition 1

1. *The correction capability of a Gabidulin code* $\mathscr{G}_k(\vec{g})$ *is* $\lfloor \frac{n-k}{2} \rfloor$
2. $\mathscr{G}_k(\vec{g})^{\perp}$ *is also a Gabidulin code.*

Proposition 2

- *Let $\boldsymbol{G} \in \mathcal{M}_{k\times n}\left(\mathbb{F}_{q^m}\right)$, a generator matrix of $\mathscr{G}_k\left(\vec{g}\right)$.*
- $\boldsymbol{T} \in \mathrm{GL}_n(\mathbb{F}_q)$

Then $\boldsymbol{G}\boldsymbol{T}$ is a generator matrix of $\mathscr{G}_k\left(\vec{g}\boldsymbol{T}\right)$.

Proof.

For the proof, remark that

$$(\vec{g}\boldsymbol{T})^{q^i} = \vec{g}^{q^i}\boldsymbol{T} \quad \text{since} \quad \boldsymbol{T}^{q^i} = \boldsymbol{T}$$

for any integer i. □ □

Plan

Key generation.

- k, ℓ, n and m are some integers such that $k < n \leqslant m$ and $\ell \ll n$.
- $\boldsymbol{G} \in \mathcal{M}_{k \times n}(\mathbb{F}_{q^m})$ is a generator matrix of $\mathscr{G}_k(\vec{g})$
- Pick at random $\boldsymbol{S} \in \mathrm{GL}_k(\mathbb{F}_{q^m})$.
- Pick a random matrix $\boldsymbol{X} \in \mathcal{M}_{k \times \ell}(\mathbb{F}_{q^m})$
- Let $\boldsymbol{P} \in \mathrm{GL}_{n+\ell}(\mathbb{F}_q)$ be a random non-singular matrix
- Compute

$$\boldsymbol{G}_{pub} \stackrel{\text{def}}{=} \boldsymbol{S}(\boldsymbol{X} \mid \boldsymbol{G})\boldsymbol{P} \tag{1}$$

The public key is $(\boldsymbol{G}_{pub}, t)$ where $t \stackrel{\text{def}}{=} \lfloor \frac{n-k}{2} \rfloor$

General GPT Cryptosystem

Encryption.

To encrypt a message $\vec{m} \in \mathbb{F}_{q^m}^k$,

1. Generate $\vec{e} \in \mathbb{F}_{q^m}^n$ such that $\|\vec{e}\|_q \leqslant t$.
2. The ciphertext is the vector

$$\vec{c} = \vec{m}\boldsymbol{G}_{pub} + \vec{e}$$

Decryption.

1. Compute $\vec{z} = \vec{c}\boldsymbol{P}^{-1}$ $\qquad \vec{z} = \vec{m}\boldsymbol{S}(\boldsymbol{X} \mid \boldsymbol{G}) + \vec{e}\boldsymbol{P}^{-1}$
2. Let $\vec{z}'$ be the last n components of $\vec{z}$ $\qquad \vec{z}' = \vec{m}\boldsymbol{S}\boldsymbol{G} + \vec{e}'$
3. Compute $\vec{y} = Dec_{.\boldsymbol{G}}(\vec{z}')$ $\qquad \vec{y} = \vec{m}\boldsymbol{S}$ since $\|\vec{e}'\|_q \leqslant \|\vec{e}\|_q \leqslant t$
4. Return $\vec{m}' = \vec{y}\boldsymbol{S}^{-1}$ $\qquad \vec{m}' = \vec{m}$

General GPT Cryptosystem

Parameters

m	k	WF general decoding
48	10	2^{134}
48	16	2^{124}
48	24	2^{198}

Structural attack

- Overbeck's attack '05 '08
- Polynomial: $\mathcal{O}((n+\ell)^3)$

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Plan

Definition 6 (Distinguisher)

- f is an integer such that $f \leqslant n-k$

We define the application Λ_f by:

$$\begin{array}{rccc} \Lambda_f : & \mathcal{M}_{\ell\times n}(\mathbb{F}_{q^m}) & \longrightarrow & \mathcal{M}_{((f+1)\ell)\times n}(\mathbb{F}_{q^m}) \\ & \boldsymbol{M} & \longmapsto & \Lambda_f(\boldsymbol{M}) \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{M}^{[0]} \\ \boldsymbol{M}^{[1]} \\ \vdots \\ \boldsymbol{M}^{[f]} \end{pmatrix} = \begin{pmatrix} \boldsymbol{M}^{q^0} \\ \boldsymbol{M}^{q^1} \\ \vdots \\ \boldsymbol{M}^{q^f} \end{pmatrix} \end{array}$$

Remark 1

- *for* $\boldsymbol{G} \in \mathcal{M}_{k\times n}(\mathbb{F}_{q^m})$ *and* $\boldsymbol{P} \in \mathcal{M}_{n\times n}(\mathbb{F}_q)$

$$\Lambda_f(\boldsymbol{GP}) = \Lambda_f(\boldsymbol{G})\boldsymbol{P}$$

- *If* $\boldsymbol{S} \in \mathcal{M}_{k\times k}(\mathbb{F}_{q^m})$ *is a non singular matrix,*

$$< \Lambda_f(\boldsymbol{SG}) > = < \Lambda_f(\boldsymbol{G}) >$$

Overbeck's attack

Proposition 3

- $\boldsymbol{G}$ *is a generator matrix of* $\mathscr{G}_k(\vec{g})$
- $f \leqslant n - k - 1$

$$< \Lambda_f(\boldsymbol{G}) >= \mathscr{G}_{k+f}(\vec{g})$$

Theorem 7

For a random $\boldsymbol{M} \in \mathcal{M}_{k \times n}(\mathbb{F}_{q^m})$*, we have*

$$Rank(\Lambda_f(\boldsymbol{M})) = min\{n, k(f+1)\}$$

with a high probability.

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Proposition 4

- Let $\boldsymbol{G}_{pub} = \boldsymbol{S}\left(\boldsymbol{X} \mid \boldsymbol{G}\right)\boldsymbol{P}$
- $f = n - k - 1$

By some additional transformations on the rows of $\Lambda_f\left(\boldsymbol{G}_{pub}\right)$, *we can get:*

$$\boldsymbol{G}'_{pub} = \begin{pmatrix} \boldsymbol{X}_1 & \boldsymbol{G}_{n-1} \\ \boldsymbol{X}_2 & \boldsymbol{0} \end{pmatrix}\boldsymbol{P}$$

Where $\boldsymbol{G}_{n-1}$ *is a generator matrix of* $\mathscr{G}_{n-1}\left(\vec{g}\right)$.

Overbeck's attack

Remark 2

$$Rank\left(\Lambda_f(\boldsymbol{G}_{pub})\right) = Rank\left(\boldsymbol{G}'_{pub}\right) = n - 1 + Rank\left(\boldsymbol{X}_2\right)$$

Theorem 8

If $Rank\left(\boldsymbol{X}_2\right) = \ell$ then,

- $$\dim < \Lambda_f(\boldsymbol{G}_{pub}) >^{\perp} = 1$$
- $$< \Lambda_f(\boldsymbol{G}_{pub}) >^{\perp} = \left\{ \left(0 \mid \alpha \vec{h}\right) \left(\boldsymbol{P}^{-1}\right)^{T} \ : \ \alpha \in \mathbb{F}_{q^m} \right\}$$

$\vec{h} \in \mathbb{F}_{q^m}^{n}$ *such that* $\left\| \vec{h} \right\|_q = n$ *and* $\boldsymbol{G}\vec{h}^T = \boldsymbol{0}$

Overbeck's attack

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$\vec{h} \in \mathbb{F}_{q^m}^n$ *such that* $\left\|\vec{h}\right\|_q = n$ *and* $\boldsymbol{G}\vec{h}^T = \boldsymbol{0}$

Overbeck's attack

Summary

- $f = n - k - 1$
- Compute

$$< \Lambda_f(\boldsymbol{G}_{pub}) >^{\perp}$$

- If

$$\dim < \Lambda_f(\boldsymbol{G}_{pub}) >^{\perp} = 1$$

- Choose $\vec{h} \in < \Lambda_f(\boldsymbol{G}_{pub}) >^{\perp}$ with $\vec{h} \neq \mathbf{0}$
- Find $\boldsymbol{T} \in \mathrm{GL}_{n+\ell}(\mathbb{F}_q)$ with $\vec{h} = (\mathbf{0} \mid \vec{h}')\boldsymbol{T}$

Remark 3

The success of this attack is:

1. *Linked to the fact that the matrix* $\boldsymbol{P} \in \mathrm{GL}_{n+\ell}(\mathbb{F}_q)$ *is defined on the based field* $\mathbb{F}_q$
2. *Also based on the supposition that* $\boldsymbol{X}_2$ *is of full rank* $Rank\,(\boldsymbol{X}_2) = \ell$

GPT Reparations

Reparation ideas linked to $\boldsymbol{X}$

- **Loidreau '10** : Proposition of parameters for $\boldsymbol{X}$ such that $Rank\left(\Lambda_f(\boldsymbol{G}_{pub})\right) < n + \ell - 1$.
- **Rashwan-Gabidulin-Honary '10** : Smart approach of the GPT Cryptosystem. The authors propose a design of $\boldsymbol{X}$ such that $Rank\left(\Lambda_f(\boldsymbol{G}_{pub})\right) < n + \ell - 1$ or such that $Rank\left(\boldsymbol{X}_2\right) < \ell$

→ Attack of **[Horlemann-Trautmann, Marshall, Rosenthal]**: Extension of Overbeck's Attack for Gabidulin-based Cryptosystems, November 2015

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$\rightarrow$ Attack of **[Horlemann-Trautmann, Marshall, Rosenthal]**: Extension of Overbeck's Attack for Gabidulin-based Cryptosystems, November 2015

GPT Reparations

Reparation ideas linked to P

These variants consist to select $P \in \mathrm{GL}_{n+\ell}(\mathbb{F}_{q^m})$

- **Gabidulin ’08**
- **Rashwan-Gabidulin-Honary ’10**

Plan

Gabidulin's Variant

Key generation.

Choose $\boldsymbol{P} \in \mathrm{GL}_{n+\ell}(\mathbb{F}_{q^m})$ such that

$$\boldsymbol{P}^{-1} = \begin{pmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ \boldsymbol{Q}_{21} & \boldsymbol{Q}_{22} \end{pmatrix} \tag{2}$$

with

- $\boldsymbol{Q}_{11} \in \mathcal{M}_{\ell\times\ell}(\mathbb{F}_{q^m})$
- $\boldsymbol{Q}_{21} \in \mathcal{M}_{n\times\ell}(\mathbb{F}_{q^m})$
- $\boldsymbol{Q}_{12} \in \mathcal{M}_{\ell\times n}(\mathbb{F}_{q^m})$ so that $\mathtt{Rank}_{\mathbb{F}_q}(\boldsymbol{Q}_{12}) = s$
- $\boldsymbol{Q}_{22} \in \mathcal{M}_{n\times n}(\mathbb{F}_q)$

Compute

$$\boldsymbol{G}_{pub} \stackrel{\text{def}}{=} \boldsymbol{S}(\boldsymbol{X} \mid \boldsymbol{G})\boldsymbol{P} \tag{3}$$

The public key is $(\boldsymbol{G}_{\mathrm{pub}}, t_{\mathrm{pub}})$ where $t_{\mathrm{pub}} \stackrel{\text{def}}{=} t - s$

Plan

Lemma 9

There exist

- $\boldsymbol{P}_{11} \in \mathrm{GL}_{\ell+s}(\mathbb{F}_{q^m})$
- $\boldsymbol{P}_{22} \in \mathrm{GL}_{n-s}(\mathbb{F}_q)$
- $\boldsymbol{P}_{21} \in \mathcal{M}_{(n-s)\times(\ell+s)}(\mathbb{F}_{q^m})$
- $\boldsymbol{L}$ *and* $\boldsymbol{R}$ *belonging to* $\mathrm{GL}_n(\mathbb{F}_q)$

Such that

$$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{I}_\ell & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{L} \end{pmatrix} \begin{pmatrix} \boldsymbol{P}_{11} & \boldsymbol{0} \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{I}_\ell & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R} \end{pmatrix} \tag{4}$$

Theorem 10

There exist

- $\boldsymbol{X}^* \in \mathcal{M}_{k\times(\ell+s)}\left(\mathbb{F}_{q^m}\right)$
- $\boldsymbol{P}^* \in \mathrm{GL}_{n+\ell}\left(\mathbb{F}_q\right)$
- $\boldsymbol{G}^*$ *that defines an* $(n-s,k)-$*Gabidulin code* $\mathscr{G}_k\left(\vec{g}^*\right)$ *such that*

$$\boldsymbol{G}_{\mathrm{pub}} = \boldsymbol{S}\left(\boldsymbol{X}^* \mid \boldsymbol{G}^*\right)\boldsymbol{P}^*. \tag{5}$$

Furthermore, the error correction capability of $\mathscr{G}_k\left(\vec{g}^*\right)$ *is*

$$t^* = t - \frac{1}{2}s > t - s = t_{\mathrm{pub}}$$

Corollary 11

The system can be broken by applying Overbeck's attack on $\boldsymbol{G}_{\mathrm{pub}}$ *with*

$$f = n - s - k - 1$$

Plan

Gabidulin, Rashwan and Honary Variant

Key generation

Choose $\boldsymbol{P} \in \mathrm{GL}_n(\mathbb{F}_{q^m})$ such that

$$\boldsymbol{P}^{-1} = (\boldsymbol{Q}_1 \mid \boldsymbol{Q}_2) \tag{6}$$

where

- $\boldsymbol{Q}_1 \in \mathcal{M}_{n \times a}(\mathbb{F}_{q^m})$
- while $\boldsymbol{Q}_2 \in \mathcal{M}_{n \times (n-a)}(\mathbb{F}_q)$

- $a \stackrel{\text{def}}{=} t - t_{\text{pub}} \implies t_{\text{pub}} = t - a$

Remark 4

$$(\boldsymbol{Q}_1 \mid \boldsymbol{Q}_2) \Longrightarrow \begin{pmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ \boldsymbol{Q}_{21} & \boldsymbol{Q}_{22} \end{pmatrix} \Longrightarrow \begin{pmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ \boldsymbol{Q}_{21} & \boldsymbol{Q}_{22} \end{pmatrix} \quad \textit{with} \quad s = a$$

Corollary 12

One can recover an alternative secret key by applying Overbeck's attack with

$$f = n - a - k - 1$$

Experimental Results

m	k	t	t_{pub}	Time (second)
20	10	5	4	$\leqslant 1$
28	14	7	3	$\leqslant 1$
28	14	7	4	$\leqslant 1$
28	14	7	5	$\leqslant 1$
28	14	7	6	$\leqslant 1$
20	10	5	4	$\leqslant 1$

Table : Parameters where $n = m$ and at least 80-bit security.

Plan

Conclusion and Related Work

1. **Overbeck's attack**: Principal threat of Gabidulin-based Schemes
2. Taking $\boldsymbol{P} \in \mathrm{GL}(\mathbb{F}_{q^m})$ might protect against it
3. In practice,

$$\boldsymbol{P}^{-1} = \begin{pmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ \boldsymbol{Q}_{21} & \boldsymbol{Q}_{22} \end{pmatrix} \quad \text{with} \quad \boldsymbol{Q}_{22} \in \mathrm{GL}(\mathbb{F}_q) \quad \text{and} \quad \mathrm{Rank}_{\mathbb{F}_q}(\boldsymbol{Q}_{12}) = s$$

$\rightsquigarrow$ **Our works give a polynomial attack**

	Matrix	Code generated	Length	Correction capability
Secret	$\boldsymbol{G}$	$\mathscr{G}_k(\vec{g})$	n	t
Attack	$\boldsymbol{G}^*$	$\mathscr{G}_k(\vec{g}^*)$	$n - s$	$t - \frac{s}{2}$
Public	$\boldsymbol{G}_{\text{pub}}$	$(n+\ell, k)$−code	$n + \ell$	$t - s$

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1. **Overbeck's attack**: Principal threat of Gabidulin-based Schemes
2. Taking $\boldsymbol{P} \in \mathrm{GL}(\mathbb{F}_{q^m})$ might protect against it
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New variant from **P. Loidreau** '16

- $\mathscr{V} \subset \mathbb{F}_{q^m}$ a $\mathbb{F}_q$−vector space
- $d = \dim_{\mathbb{F}_q}(\mathscr{V}) \geqslant 3$
- Choose

$$\boldsymbol{P}^{-1} \in \mathrm{GL}_n(\mathscr{V}) \quad \text{and} \quad \boldsymbol{G}_{\mathrm{pub}} = \boldsymbol{SGP}$$

$$\rightarrow \color{red}{t_{\mathrm{pub}}} = \frac{n-k}{2\color{red}{d}}$$